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Confining Potential in Momentum Space

by

John W. Norbury[1]
Department of Physics
Rider College
Lawrenceville, NJ 08648

David E. Kahana[2]
Continuous Electron Beam Accelerator Facility
Newport News, VA 23606

Khin Maung Maung[3]
Department of Physics
Hampton University
Hampton, Virginia 23668
and
NASA Langley Research Center
Hampton, Virginia 23665

[1]Supported by NASA grant # NAG 1-1134
[2]DOE funded research at CEBAF
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Abstract

A method is presented for solution in momentum space of the bound state problem with linear potential in r–space. The potential is unbounded at large r leading to a singularity at small q. The singularity is integrable, when regulated by exponentially screening the r–space potential, and is removed by a subtraction technique. The limit of zero screening is taken analytically, and numerical solution of the subtracted integral equation gives eigenvalues and wavefunctions in good agreement with position space calculations. The method generalises easily to arbitrary power law potentials.

Lattice gauge calculations[1] for static (heavy) quarks support the notion that the interquark potential in QCD behaves as $V(r) \sim \lambda r$ for large r. Indeed, the linear potential has long been used in phenomenological non-relativistic quark models of baryons and mesons[2,3]. Meson spectroscopy in particular is successfully described by a linear potential at large r, modified by spin and colour dependent Coulomb forces at small r. Most calculations with the linear potential are carried out in coordinate space. This is the simplest procedure for heavy quark systems, which can perhaps be considered as non-relativistic; however for light quark systems it would be desirable to have a relativistic treatment. Bound state equations in relativistic systems[4] are generally much easier to solve in momentum space, and thus we are led to consider, as a starting point for the relativistic case, the Lippman–Schwinger equation for two scalar particles interacting by a linear potential. The methods developed will generalise relatively straightforwardly to relativistic treatments.

To summarise: here, we treat the Lippmann-Schwinger equation for a linear r-space potential. The method is for the most part straightforward, the only difficulty arising from the singularity of the kernel at the origin of momentum space. So far as we are aware, previous studies of the linear potential in momentum space[5] have been approximate, in the sense that the singularity was handled by screening the r space potential:

$$V(r) \sim \lambda r \, e^{-\eta r}. \tag{1}$$

What has perhaps not been generally appreciated is that the limit $\eta \rightarrow 0$ can be taken analytically. To the best of our knowledge, previous treatments keep the parameter η finite, leading to some uncertainty as to the nature of the calculated eigenvalues and wavefunctions. In this connection, recall that the screened linear potential does not strictly speaking possess true bound states, instead it has scattering resonances, which for low energy approximate bound states of the unscreened potential. We will extract the limit of zero screening analytically, using a subtraction technique. The resulting subtracted integral equation is relatively easy to handle numerically.

The Lippmann–Schwinger equation for the l^{th} partial wave is

$$\frac{p^2}{2\mu}\phi_l(p) + \int V_l(p,p')\phi_l(p')p'^2 dp' = E\phi_l(p) \tag{2}$$

The limit $\eta \to 0$ now exists, and may be extracted by splitting the region of integration to isolate the singularity. We write:

$$\begin{aligned}\int dp' \left[Q_0'(y) + \frac{\eta^2}{pp'}Q_0''(y)\right](\phi_0(p') - \phi_0(p)) &= \int_0^{p-4\eta} + \int_{p-4\eta}^{p+4\eta} + \int_{p+4\eta}^{\infty} \\ &= A \quad + \quad B \quad + \quad C\end{aligned} \tag{8}$$

The limits $p \pm 4\eta$ have been chosen so that all three extrema of the kernel lie in the middle region B. The explicit forms of the Legendre functions are:

$$Q_0'(y) = \frac{1}{1-y^2} = pp'\left[\frac{-1}{(p'-p)^2+\eta^2} + \frac{1}{(p'+p)^2+\eta^2}\right]$$

and

$$\frac{\eta^2}{pp'}Q_0''(y) = \eta^2(p^2+p'^2+\eta^2)\left[\frac{-1}{(p'-p)^2+\eta^2} + \frac{1}{(p'+p)^2+\eta^2}\right]^2.$$

It is clear that for $p' \neq p$, as is the case in the integrals A and C, the limit $\eta \to 0$ is innocuous, and may be taken immediately, indeed one has:

$$\lim_{\eta\to 0}[A+C] = P\int_0^\infty dp'\left[\frac{-4p^2p'^2}{(p'^2-p^2)^2}\right][\phi_0(p') - \phi_0(p)] \tag{9}$$

where P denotes as usual, the Cauchy principal value of the integral, which has been made well-defined by the subtraction. The term B must be handled with care, however, since $p' = p$ inside the region of integration. Assuming $\phi(p')$ is analytic in the neighborhood of p, and making an obvious change of variable we find:

$$\begin{aligned}\lim_{\eta\to 0} B = \lim_{\eta\to 0}\int_{-4\eta}^{4\eta} dx \Bigg\{&\left[p(p+x)\left[\frac{-1}{x^2+\eta^2} + \frac{1}{(x+2p)^2+\eta^2}\right]\left[x\phi' + \frac{x^2}{2}\phi'' + \cdots\right]\right] \\ &+ \left[\eta^2((x+p)^2+p^2+\eta^2)\left[\frac{-1}{x^2+\eta^2} + \frac{1}{(x+2p)^2+\eta^2}\right]^2\left[x\phi' + \frac{x^2}{2}\phi'' + \cdots\right]\right]\Bigg\} \\ &= \lim_{\eta\to 0} B1 + \lim_{\eta\to 0} B2\end{aligned} \tag{10}$$

Scaling out 4η then results in:

$$\begin{aligned}\lim_{\eta\to 0} B1 &= \lim_{\eta\to 0}\int_{-1}^{1}(4\eta)\,dy\,\frac{p}{4\eta}\left(\frac{p}{4\eta}+y\right)\left(\frac{-1}{1+y^2}\right)\left[(4\eta)y\phi' + \frac{(4\eta)^2y^2}{2}\phi'' + \cdots\right] \\ &= (p^2\phi'(p))\int_{-1}^{1}dy\left(\frac{-y}{1+y^2}\right) = 0.\end{aligned}$$

(11)

The contribution of the second term in $B1$ clearly vanishes since it is not singular at $p' = p$, the analysis of $B2$ is similar, and we conclude that B tends to zero. Therefore the limiting form of the equation is:

$$\frac{p^2}{2\mu}\phi_0(p) - \frac{\lambda}{\pi p^2} P \int_0^\infty dp' \left[\frac{4p^2p'^2}{(p'^2-p^2)^2}\right] (\phi_0(p') - \phi_0(p)) = E\phi_0(p) \tag{12}$$

We now discuss the numerical solution of eq(12), which is not yet a completely trivial matter, since care must be taken to obtain the Cauchy principal value. In this respect there is a difference between the linear potential and the Coulomb potential, the latter giving rise to a logarithmic singularity. For the Coulomb potential, the method used in the literature[6] is directly to write the Coulomb analog of eq(12), for example using Gaussian quadrature, as a matrix equation. Since the singularity is only logarithmic this method is successful for the Coulomb potential. Here, such an approach is not feasible. Instead, we expand ϕ_0 in a suitable set of basis functions:

$$\phi_0(p) = \sum_n^N C_n g_n(p) \tag{13}$$

Inserting this expansion in eq(12), multiplying by $p^2 g_m(p)$ and integrating over p, we obtain:

$$\sum_n C_n \{ \int \frac{p^4}{2\mu} g_m(p) g_n(p) dp + \frac{\lambda}{\pi} \int \left[\frac{4p^2p'^2}{(p'^2-p^2)^2}\right] g_m(p) [g_n(p') - g_n(p)] dp' dp$$

$$= E \sum_n C_n \int p^2 g_m(p) g_n(p) dp \tag{14}$$

which is just the matrix equation:

$$\sum_n A_{mn} C_n = E \sum_n G_{mn} C_n \tag{15}$$

The double integral over p and p' is performed by changing to variables $(p'+p)$ and $(p'-p)$. The singularity is in the integral over $(p'-p)$, so this is carried out first using Gaussian quadrature with an even number of points. This type of integration yields the Cauchy principal value automatically[7]. A convenient set of functions $g(p)$ is:

$$g_n(p) = \frac{1}{(n^2/N)^2 + p^4} \tag{16}$$

where N is the maximum number of functions used in the expansion eq(13). Fig(2) is a 3-d plot of the kernel of eq(14), showing clearly the cancellation which leads to the principal value. Using the above method we have calculated both eigenvalues and eigenvectors. In table I the first 12 eigenvalues are listed. We used $m_1 = m_2 = 1.5\ GeV$ and the string tension $\lambda = 5\ (GeV)^2$. One can see that the lower eigenvalues converge nicely as the number of functions is increased. We compare to the eigenvalues obtained from a coordinate space calculation (integrating the equation out from $r = 0$ and in from large r, and matching the logarithmic derivatives at the classical turning point), in table (1). The calculated eigenfunctions also agree with the coordinate space calculation.

In conclusion, we have treated the problem of two non-relativistic, scalar particles interacting via a linear potential in momentum space. The relevant Lippmann–Schwinger equation has a singular kernel. We have shown how after regulating the singularity by exponentially screening the r-space potential, the severity of the singularity can be reduced by a suitable subtraction, and the limit of zero screening extracted analytically. To the best of our knowledge, this point has not been generally understood in the literature. The limiting form of the equation has been treated numerically, and results are in good agreement with more straightforward coordinate space calculations. Relativistic equations involving linear potentials involve similar singularities, so that the methods developed here will be applicable. We intend to study the relativistic quark-antiquark problem in the future. The method presented here can be generalised to arbitrary power law potentials, and to higher partial waves without undue difficulty.

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Table I

Energy eigenvalues in GeV for $l = 0$, $m_1 = m_2 = 1.5$ GeV and $\lambda = 5\ (GeV)^2$

$N =$	8	10	12	14	16	18	Exact
E_1	5.973	5.972	5.972	5.972	5.972	5.972	5.971
E_2	10.468	10.444	10.443	10.443	10.443	10.443	10.441
E_3	14.389	14.114	14.111	14.104	14.104	14.104	14.101
E_4	18.646	17.452	17.378	17.341	17.335	17.335	17.335
E_5	23.402	21.125	20.397	20.351	20.294	20.293	20.291
E_6	27.206	25.683	23.440	23.281	23.072	23.053	23.046
E_7	33.032	31.269	27.274	26.059	25.842	25.648	25.646
E_8	44.374	36.224	32.113	29.032	28.789	27.947	28.119
E_9		40.519	38.146	33.051	31.561	30.194	30.488
E_{10}		51.774	45.309	38.067	34.428	33.340	32.769
E_{11}			49.940	44.286	38.517	36.489	34.972
E_{12}			58.588	51.893	43.615	37.309	37.109

Figure Captions

Fig. 1

The singularity structure of the kernel is shown for finite $\eta = .075$ with fixed $p = 2$.

Fig. 2

A 3-dimensional figure of the subtracted, regulated integrand; $\eta = .075$. The cancellation which produces the Cauchy principle value is evident.

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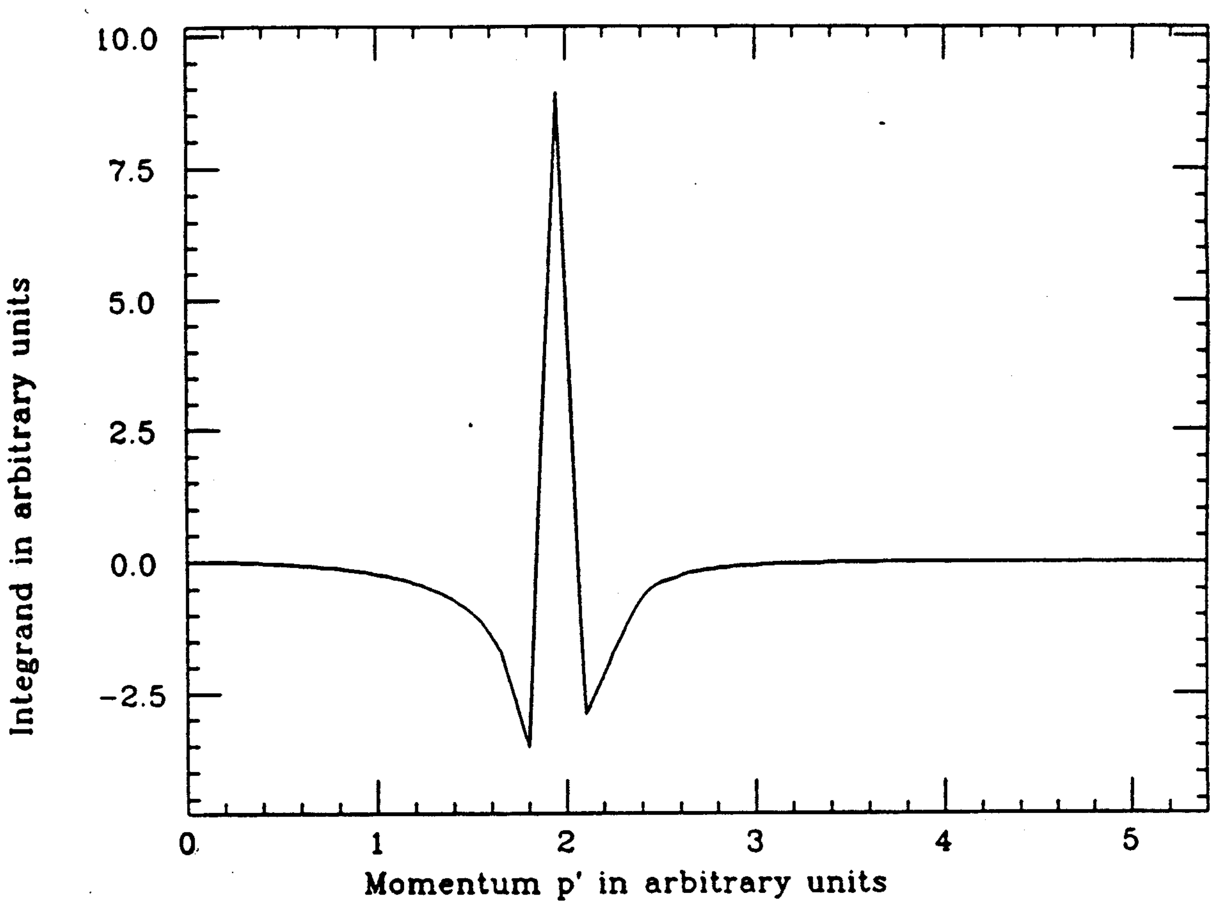

10.0
7.5
5.0
2.5
0.0
-2.5
Integrand in arbitrary units
0
1
2
3
4
5
Momentum p' in arbitrary units

Fig. 1

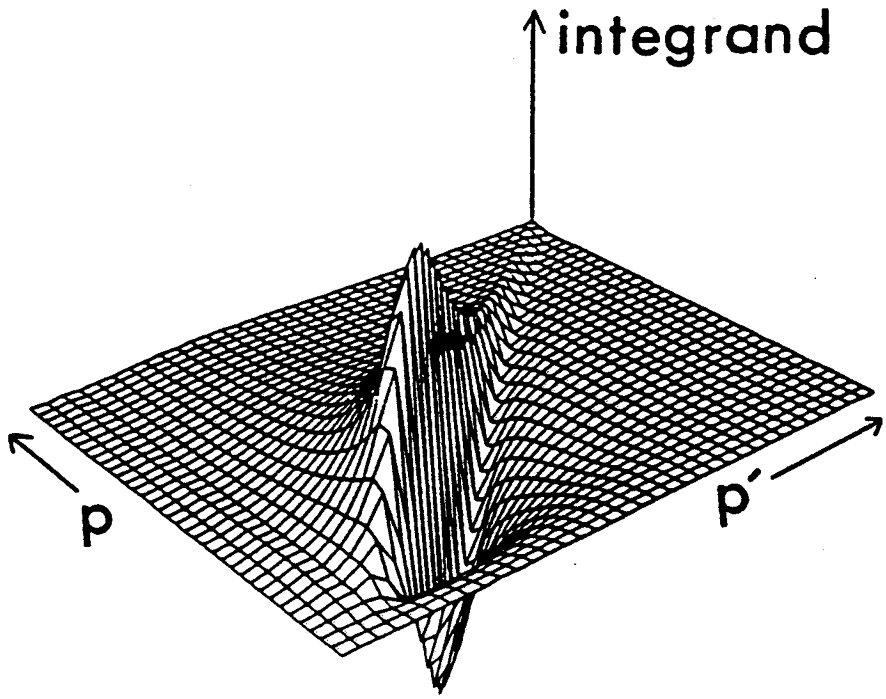

Fig. 2